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LETTER TO THE EDITOR

Height probabilities in solid-on-solid models: I

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Abstract. The height probabilities for some infinite sequences of solid-on-solid models are given in terms of combinatorial sums for the large but finite lattice. We transform a class of these sums to a form suitable for taking the infinite lattice limit.

The year 1984 saw some exciting events in the study of multicritical phenomena in two dimensions. The conformal bootstrap theory of Belavin *et al* (1984) was developed by Friedan *et al* (1984) to provide a listing of allowed critical exponents η for generic multicritical behaviour. At about the same time, Andrews *et al* (1984) exactly evaluated the free energy and one-point functions for an infinite sequence of restricted solid-on-solid (RSOS) models. Realising the relationship between the two works, it was observed by Huse (1984) that the critical exponents of the RSOS model provide an explicit realisation of the critical exponents allowed by conformal invariance for all generic n-phase coexistence.

It is known that, corresponding to a particular central charge in the conformal theories, there correspond many possible operator algebras, so the listing of allowed critical exponents is not complete (Cardy 1986). Likewise, it is now known that the RSOS model is just one of an infinite number of infinite sequences of solvable models (Kuniba *et al* 1986). It is our purpose in this letter to initiate the evaluation of the local height probabilities for these new hierarchies.

Consider the following class of interaction-round-a-face models. At each site j of the square lattice there is an integer height variable l_j, $1 \leq l_j \leq r-1$. For a given integer n ($n \geq 2$), impose the constraint that nearest-neighbour heights must differ by

$$\begin{aligned} &0, \pm 1, \pm 2, \ldots, \pm\tfrac{1}{2}(n-1) \qquad && (n \text{ odd}) \\ &\pm 1, \pm 2, \ldots, \pm\tfrac{1}{2}n && (n \text{ even}). \end{aligned} \tag{1}$$

We will denote such sequences of models by RSOS-n.

For each value of $r \geq 4$ the RSOS-2 model has an exact solution manifold, on which the Boltzmann weights of the allowed face configurations are parametrised by single θ_1 functions (Andrews *et al* 1984). In the limits $r \to \infty$, $l_j \to \infty$, the Boltzmann weights become sinh functions and we obtain the six-vertex (two-state vertex) parametrisation of the body-centred solid-on-solid model (Forrester 1986).

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The RSOS-2 model has different regimes of physical behaviour, denoted by regimes I-IV. In regime III, calculation of the height probabilities on a lattice of size m using the corner transfer matrix technique yields the combinatorial sum

$$ {}_nX_m(a, b, c) = \sum_{l_2, \ldots, l_m = 1}^{r-1} q^{\sum_{k=1}^{m} k|l_k - l_{k+2}|/4} \tag{2}$$

where $l_1, l_2, \ldots, l_{m+2}$ must satisfy the nearest-neighbour constraint (1) for $n = 2$ (equations (1.5.11)-(1.5.15) of Andrews *et al* (1984)). The end heights l_1, l_{m+1}, l_{m+2} are fixed at the values

$$ l_1 = a \qquad l_{m+1} = b \qquad l_{m+2} = c. \tag{3}$$

It has been shown recently by Kuniba *et al* (1986) that the RSOS-3 model admits an exact solution manifold for $r \geqslant 3$. The weights are parametrised as products of two θ_1 functions. We observe that in the limits $r \to \infty$, $l \to \infty$ this parametrisation reduces to the three-state vertex parametrisation of Sogo *et al* (1983). (Explicitly, if $p = e^{-\varepsilon}$ is the nome of the θ_1 function and u the argument, we consider the limit $1/r\varepsilon$, $u/\varepsilon \to$ constant, $1/r$, u, $\varepsilon \to 0$.) Again we have different regimes of physical behaviour to consider. In regime III, by applying the corner transfer matrix technique, we can show that the local height probabilities can be written in terms of the sum (2) for $n = 3$.

Kuniba *et al* (1986) conjecture a solvable infinite sequence of RSOS-n models for each n. This is very plausible, since in the limit $r \to \infty$, $l \to \infty$, we regain the n-state vertex model of Sogo *et al* (1983), where it is known there is a solvable manifold for each n. We conjecture that in regime III of the exactly solvable RSOS-n model, calculation of the height probabilities from the corner transfer matrix technique gives the combinatorial sum ${}_nX_m$. For $n = 4$, this can be explicitly verified in the case $r \to \infty$ from the parametrisation of the four-state vertex weights by Sogo *et al* (1983).

Let us sketch the derivation of the transformation formula necessary to extract the $m \to \infty$ limit from (2) in the case $n = 3$.

We begin by re-analysing our results for the RSOS-2 hierarchy (equations (2.3.5) and (2.3.6) of Andrews *et al* (1984)). In the limit $r \to \infty$, then $a, b \to \infty$, $a - b =$ constant, the sum (${}_2X_m^*$ say) can be written as a single Gaussian polynomial:

$$ {}_2X_m^*(a, b, b \pm 1) = q^{(b-a)(b-a\pm 1)/4} \begin{bmatrix} m \\ (m + a - b)/2 \end{bmatrix}. \tag{4}$$

We note that ${}_2X_m^*$ can be specified by defining what we will call the sum index (SI) of the permutations of a set. Consider a permutation P of $\{1^{m_1}(-1)^{m_2}\}$ (in this set there are m_1 elements 1 and m_2 elements -1). Let $\mathrm{SI}(m_1, m_2; n)$ denote the number of permutations P for which

$$ \sum_{j=1}^{m_1+m_2} j \frac{|P(j) + P(j+1)|}{2} = n \tag{5}$$

where $P(m_1 + m_2 + 1) = \pm 1$ according to whether we take the plus or minus sign in (4). Then

$$ {}_2X_m^*(a, b, b \pm 1) = \sum_{n \geqslant 0} \mathrm{SI}\left(\frac{m + b - a}{2}, \frac{m - b + a}{2}; n\right) q^{n/2}. \tag{6}$$

Comparing (4) and (6) we see that up to a simple factor, the Gaussian polynomial is the generating function for the sum index. This result is analogous to that for the greater index of the set $\{1^{m_1}(-1)^{m_2}\}$ (Andrews 1976, theorem 3.7).

Now consider ${}_3X_m(a, b, b \pm 1)$ in the limit $r \to \infty$, then $a, b \to \infty$, $a - b = \text{constant}$ (let us denote this sum ${}_3X^*_m$). Define the sum index $\mathrm{SI}(m_1, m_2, m_3; n)$ of the permutations P of $\{1^{m_1}(-1)^{m_2}0^{m_3}\}$ by (5) with $m_1 + m_2$ replaced by $m_1 + m_2 + m_3$ and $P(m_1 + m_2 + m_3 + 1) = \pm 1$ according to whether we take the plus or minus sign in ${}_3X^*_m(a, b, b \pm 1)$. Then

$$ {}_3X^*_m(a, b, b \pm 1) = \sum_{\mu \geqslant 0} \sum_{n \geqslant 0} \mathrm{SI}(\mu + b - a, \mu, m - b + a - 2\mu; \tfrac{1}{2}n) q^{n/4} \tag{7} $$

where the upper limit on the μ summation is the integer part of $(m - b + a)/2$ and we have taken $b \geqslant a$.

Consider the sum over n in (7). If the sum index were the greater index, we would be able to express it in terms of the q-multinomial coefficient with with three indices (the Gaussian polynomial is the q-multinomial coefficient with two indices). We have just seen that the sum and greater indices are closely related. We thus set out to express the sum over n as a single multinomial coefficient. Working empirically, we found instead the result

$$ \sum_{n \geqslant 0} \mathrm{SI}(\mu + b - a, \mu, m - b + a - 2\mu; \tfrac{1}{2}n) q^{n/4} $$

$$ = q^{(b-a)(b-a\pm 1)/4} q^{(m-b+a-2\mu)/4} \begin{bmatrix} m \\ 2\mu + b - a \end{bmatrix}_{q^{1/2}} \begin{bmatrix} 2\mu + b - a \\ \mu \end{bmatrix}_q . \tag{8} $$

(If the variables of the Gaussian polynomials were both the same, this would be, up to a simple factor, a multinomial coefficient.) Note that when $2\mu = m - b + a$, and thus the total number of zeros is zero, we regain (4).

From here, guided by the similarity between ${}_2X_m$ and ${}_2X^*_m$ we make an ansatz for ${}_3X^*_m$ based on the form (7) and (8) for ${}_3X^*_m$. This ansatz is used in conjunction with the recurrence relations, initial and boundary conditions which uniquely define the ${}_nX_m$ (the analogues of equations (2.3.1)-(2.3.4) of Andrews *et al* (1984)).

We thus obtain the transformation formulae

$$ {}_3X_m(a, b, b \pm 1) = q^{a(a-1)/4}(g(a, b, b \pm 1) - g(-a, b, b \pm 1)) \tag{9} $$

and

$$ \begin{aligned} {}_3X_m(a, b, b) = \delta_{a,b} + q^{1/4 + a(a-1)/4}\{&h(a, b+1, b) - h(-a, b+1, b) \\ &+ h(a, b-1, b) - h(-a, b-1, b)\} \end{aligned} \tag{10} $$

where

$$ g(a, b, b \pm 1) = \sum_{\lambda=-\infty}^{\infty} q^{r(r-1)\lambda^2 + \alpha(a,b,b\pm 1)\lambda + \beta(a,b,b\pm 1)} \sum_{\mu=(a-|a|)/2}^{\infty} q^{(m-|b-a|-2\mu)/4} \times \begin{bmatrix} m \\ 2\mu + |b-a| \end{bmatrix}_{q^{1/2}} \begin{bmatrix} 2\mu + |b-a| \\ \mu - r\lambda \end{bmatrix}_q \tag{11} $$

$$ h(a, b, b \pm 1) = \sum_{\lambda=-\infty}^{\infty} q^{r(r-1)\lambda^2 + \alpha(a,b,b\pm 1)\lambda + \beta(a,b,b\pm 1)} \sum_{\mu=(a-|a|)/2}^{\infty} q^{(2\mu+|b-a|)/4} \times \begin{bmatrix} m \\ 2\mu + |b-a| + 1 \end{bmatrix}_{q^{1/2}} \begin{bmatrix} 2\mu + |b-a| \\ \mu - r\lambda \end{bmatrix}_q \tag{12} $$

$$ \alpha(a, b, b \pm 1) = \mathrm{sgn}(b - a)\{r(b + b \pm 1 - 1)/2 - a(r - 1)\} \tag{13} $$

$$ \beta(a, b, b + 1) = b(b + 1)/4 - ab/2 \tag{14} $$

$$ \beta(a, b, b - 1) = b(b - 1)/4 - (b - 1)a/2. \tag{15} $$

In (10) $\delta_{a,b}$ denotes the Kronecker delta and in (13) the function sgn is defined by

$$\operatorname{sgn}(b-a)=\begin{cases} 1 & b\geqslant a \\ -1 & b<a. \end{cases}$$

Note that when $\mu=(m-|b-a|)/2$ and $\mu=(m-|b+a|)/2$ in $g(a, b, b\pm 1)$ and $g(-a, b, b\pm 1)$ respectively, (9) reduces to ${}_2X_m(a, b, b\pm 1)$ (equations (2.3.5) and (2.3.6) of Andrews *et al* (1984)).

It remains as a major mathematical project to transform ${}_nX_m$ for general n and study the large-m limit. We are pursuing this line of research.

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